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Second-order differential invariants of a family of diffusion equations

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Abstract

An equivalence transformation algebra $L_{\mathcal{E}}$ for a class of nonlinear diffusion equations is found. After having obtained the second-order differential invariants with respect to $L_{\mathcal{E}}$, we get some results which allow us to linearize a subclass of the equations considered.

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1. Introduction

Here, we consider the following diffusion equations:

$$u_t - u_{xx} = f(u, u_x), \tag{1.1}$$

which arise in several problems of mathematical physics.

By using the Lie criterion of infinitesimal invariance [1], we construct the Lie algebra $L_{\mathcal{E}}$ of the equivalence transformations. These transformations have the property to change any element of a family of PDEs to a PDE which is also a member of the same family. An equivalence transformation maps solutions of an equation of the family to solutions of the transformed equation.

Following the method proposed by Ibragimov in [2, 3] and successively applied in [4], we calculate the differential invariants with respect to the equivalence transformations of the family (1.1).

Starting from these results, we characterize a subclass of equations (1.1) which can be linearized through an equivalence transformation.

The outline of the paper is as follows. In section 2, we obtain the infinitesimal equivalence generator of equations (1.1). In section 3, we look for differential invariants and by following the infinitesimal method [2, 3], we found the second-order differential invariants for the family of equations (1.1).

Finally, in section 4, results from the previous section are used in order to characterize a subclass of the family (1.1) which can be mapped, by an equivalence transformation, in the Fourier's equation. The conclusions are given in section 5.

2. Equivalence algebra

We recall that a transformation of the type

$$t = t(\tau, \sigma, \upsilon), \qquad x = x(\tau, \sigma, \upsilon), \qquad u = u(\tau, \sigma, \upsilon),$$

which is locally a C^∞-diffeomorphism and changes the original equation into a new equation having the same differential structure but a different form of the function f, is an equivalence transformation [1] (hereafter ET) for equations (1.1). An invariance transformation can be regarded as a particular ET such that the transformed function f has the same form. In the following we consider only continuous groups of equivalence transformations.

The direct search for equivalence transformations through the finite form of the transformation is connected with considerable computational difficulties. The use of the Lie infinitesimal criterion, in the way suggested by Ovsiannikov [1], gives an algorithm to find infinitesimal generators of the ETs that overcome these problems.

In order to obtain continuous groups of ETs of equations (1.1), we consider, by following [1], the arbitrary function f as a dependent variable and apply the Lie infinitesimal invariance criterion to the following system:

$$u_t - u_{xx} = f, \qquad f_t = f_x = f_{u_t} = 0, \tag{2.1}$$

where the last three equations of (2.1) are usually called *auxiliary equations* and give the independence of f on t, x and u_t.

The infinitesimal equivalence generator Y has the following form:

$$Y = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial u_t} + \zeta_2 \frac{\partial}{\partial u_x} + \mu \frac{\partial}{\partial f}, \tag{2.2}$$

where ξ^1, ξ^2 and η are sought depending on t, x and u, while μ depends on t, x, u, u_t, u_x and f, the components ζ_1 and ζ_2, as known, are given by

$$\zeta_1 = D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \qquad \zeta_2 = D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2).$$

The operators D_t and D_x denote the total derivatives with respect to t and x:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots,$$
$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots.$$

The prolongation of operator (2.2), which we need in order to require the invariance of (2.1), is

$$\widetilde{Y} = Y + \zeta_{22} \frac{\partial}{\partial u_{xx}} + \omega_t \frac{\partial}{\partial f_t} + \omega_x \frac{\partial}{\partial f_x} + \omega_{u_t} \frac{\partial}{\partial f_{u_t}}, \tag{2.3}$$

where (see, e.g., [4, 5])

$$\zeta_{22} = D_x(\zeta_2) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2), \qquad \omega_t = \widetilde{D}_t(\mu) - f_u \widetilde{D}_t(\eta) - f_{u_x} \widetilde{D}_t(\zeta_2),$$
$$\omega_x = \widetilde{D}_x(\mu) - f_u \widetilde{D}_x(\eta) - f_{u_x} \widetilde{D}_x(\zeta_2), \qquad \omega_{u_t} = \widetilde{D}_{u_t}(\mu) - f_u \widetilde{D}_{u_t}(\eta) - f_{u_x} \widetilde{D}_{u_t}(\zeta_2),$$

while $\widetilde{D}_t$, $\widetilde{D}_x$ and $\widetilde{D}_{u_t}$ are defined by

$$\widetilde{D}_t = \frac{\partial}{\partial t} + f_t\frac{\partial}{\partial f} + f_{tt}\frac{\partial}{\partial f_t} + f_{tx}\frac{\partial}{\partial f_x} + \cdots,$$
$$\widetilde{D}_x = \frac{\partial}{\partial x} + f_x\frac{\partial}{\partial f} + f_{tx}\frac{\partial}{\partial f_t} + f_{xx}\frac{\partial}{\partial f_x} + \cdots,$$
$$\widetilde{D}_{u_t} = \frac{\partial}{\partial u_t} + f_{u_t}\frac{\partial}{\partial f} + f_{tu_t}\frac{\partial}{\partial f_t} + f_{xu_t}\frac{\partial}{\partial f_x} + \cdots.$$

Applying the operator (2.3) to the system (2.1) and following the well-known algorithm (see, e.g., [5, 6]) we obtain

$$Y = (c_0 + c_1 t)\frac{\partial}{\partial t} + \left(\frac{1}{2}c_1 x + c_2 t + c_3\right)\frac{\partial}{\partial x} + \varphi(u)\frac{\partial}{\partial u} + (-c_1 u_t - c_2 u_x + \varphi' u_t)\frac{\partial}{\partial u_t} + \left(-\frac{1}{2}c_1 u_x + \varphi' u_x\right)\frac{\partial}{\partial u_x} + \left(-c_1 f - c_2 u_x + \varphi' f - \varphi'' u_x^2\right)\frac{\partial}{\partial f},$$

where c_0, c_1, c_2 and c_3 are arbitrary constants, φ is an arbitrary function of u and the prime denotes the differentiation with respect to u. So, we have found that the Lie algebra $L_\mathcal{E}$ for the class of equations (1.1) is infinite dimensional and generates an infinite continuous group $G_\mathcal{E}$ of equivalence transformations spanned by the following operators:

$$Y_0 = \frac{\partial}{\partial t}, \qquad Y_1 = t\frac{\partial}{\partial t} + \frac{1}{2}x\frac{\partial}{\partial x} - f\frac{\partial}{\partial f} - u_t\frac{\partial}{\partial u_t} - \frac{1}{2}u_x\frac{\partial}{\partial u_x}, \tag{2.4}$$

$$Y_2 = t\frac{\partial}{\partial x} - u_x\frac{\partial}{\partial f} - u_x\frac{\partial}{\partial u_t}, \qquad Y_3 = \frac{\partial}{\partial x}, \tag{2.5}$$

$$Y_\varphi = \varphi\frac{\partial}{\partial u} + \left(\varphi' f - \varphi'' u_x^2\right)\frac{\partial}{\partial f} + \varphi' u_t\frac{\partial}{\partial u_t} + \varphi' u_x\frac{\partial}{\partial u_x}. \tag{2.6}$$

3. Search for differential invariants

Following [2–4] (see also [7]) we recall that, for the family of equations (1.1), a *differential invariant of order s* is a function J, of the independent variables t, x, the dependent variable u and its derivatives u_t, u_x, as well as of the function f and its derivatives of maximal order s, invariant with respect to the equivalence group $G_\mathcal{E}$.

Here we wish to look for all invariants J that depend on the derivatives of f up to the second order, inclusive. So, we need the following second prolongation of operator Y:

$$Y^{(2)} = Y^{(1)} + \omega_{uu}\frac{\partial}{\partial f_{uu}} + \omega_{uu_x}\frac{\partial}{\partial f_{uu_x}} + \omega_{u_xu_x}\frac{\partial}{\partial f_{u_xu_x}}$$

where [4, 5]

$$Y^{(1)} = Y + \omega_u\frac{\partial}{\partial f_u} + \omega_{u_x}\frac{\partial}{\partial f_{u_x}}$$
$$= Y + \left(-c_1 f_u + \varphi'' f - \varphi'' u_x f_{u_x} - \varphi''' u_x^2\right)\frac{\partial}{\partial f_u} - \left(\frac{1}{2}c_1 f_{u_x} + c_2 + 2\varphi'' u_x\right)\frac{\partial}{\partial f_{u_x}}$$

and

$$\omega_{uu} = \widetilde{D}_u(\omega_u) - f_{uu}\widetilde{D}_u(\eta) - f_{uu_x}\widetilde{D}_u(\zeta_2),$$
$$\omega_{uu_x} = \widetilde{D}_{u_x}(\omega_u) - f_{uu}\widetilde{D}_{u_x}(\eta) - f_{uu_x}\widetilde{D}_{u_x}(\zeta_2),$$
$$\omega_{u_xu_x} = \widetilde{D}_{u_x}(\omega_{u_x}) - f_{uu_x}\widetilde{D}_{u_x}(\eta) - f_{u_xu_x}\widetilde{D}_{u_x}(\zeta_2).$$

After some calculations we get

$$Y^{(2)} = Y^{(1)} - \left(\varphi' f_{u_x u_x} + 2\varphi''\right) \frac{\partial}{\partial f_{u_x u_x}} - \left(\frac{1}{2} c_1 f_{uu_x} + \varphi' f_{uu_x} + \varphi'' u_x f_{u_x u_x} - 2\varphi''' u_x\right) \frac{\partial}{\partial f_{uu_x}}$$
$$+ \left[-c_1 f_{uu} - \varphi' f_{uu} + \varphi''\left(f_u - 2u_x f_{uu_x}\right) + \varphi'''\left(f - u_x f_{u_x}\right) - \varphi^{IV} u_x^2\right] \frac{\partial}{\partial f_{uu}}.$$

Now by observing that

$$Y_0^{(2)} = Y_0, \qquad Y_3^{(2)} = Y_3, \qquad \hat{Y}_\varphi^{(2)} = \hat{Y}_\varphi$$

and by using the fact that the function $\varphi(u)$ and all its derivatives are arbitrary, we must look for invariant functions of the form

$$J = J\left(u_t, u_x, f, f_u, f_{u_x}, f_{uu}, f_{uu_x}, f_{u_x u_x}\right)$$

which are invariant with respect to the following operators:

$$Y_1^{(2)} = Y_1^{(1)} - f_{uu} \frac{\partial}{\partial f_{uu}} - \frac{1}{2} f_{uu_x} \frac{\partial}{\partial f_{uu_x}}, \qquad Y_2^{(2)} = Y_2^{(1)},$$
$$\hat{Y}_{\varphi'}^{(2)} = Y_{\varphi'}^{(1)} - f_{uu} \frac{\partial}{\partial f_{uu}} - f_{uu_x} \frac{\partial}{\partial f_{uu_x}} - f_{u_x u_x} \frac{\partial}{\partial f_{u_x u_x}},$$
$$\hat{Y}_{\varphi''}^{(2)} = Y_{\varphi''}^{(1)} + \left(f_u - 2u_x f_{uu_x}\right) \frac{\partial}{\partial f_{uu}} - u_x f_{u_x u_x} \frac{\partial}{\partial f_{uu_x}} - 2 \frac{\partial}{\partial f_{u_x u_x}},$$
$$\hat{Y}_{\varphi'''}^{(2)} = Y_{\varphi'''}^{(1)} + \left(f - u_x f_{u_x}\right) \frac{\partial}{\partial f_{uu}} - 2u_x \frac{\partial}{\partial f_{uu_x}}, \qquad \hat{Y}_{\varphi^{IV}}^{(2)} = -u_x^2 \frac{\partial}{\partial f_{uu}}.$$

The invariance condition $Y_{\varphi^{IV}}^{(2)}(J) = 0$ implies

$$J = J\left(u_t, u_x, f, f_u, f_{u_x}, f_{uu_x}, f_{u_x u_x}\right)$$

while $Y_{\varphi'''}^{(2)}(J) = 0$ yields

$$J = J\left(u_t, u_x, f, f_{u_x}, f_{u_x u_x}, p_1\right),$$

where

$$p_1 = \frac{f_u}{u_x} - \frac{f_{uu_x}}{2}.$$

Likewise, from $Y_2^{(2)}(J) = 0$ we obtain

$$J = J\left(u_x, f_{u_x u_x}, p_1, p_2, p_3\right),$$

where

$$p_2 = f - u_t, \qquad p_3 = \frac{f}{u_x} - f_{u_x},$$

and from $\hat{Y}_{\varphi'}^{(2)}(J) = 0$,

$$J = J(q_1, q_2, q_3, q_4), \tag{3.1}$$

where

$$q_1 = f_{u_x u_x}(f - u_t), \qquad q_2 = \left(\frac{f_u}{u_x} - \frac{f_{uu_x}}{2}\right)(f - u_t), \tag{3.2}$$

$$q_3 = p_3 = \frac{f}{u_x} - f_{u_x}, \qquad q_4 = u_x f_{u_x u_x}. \tag{3.3}$$

By applying operator $\hat{Y}^{(2)}_{\varphi''}$ to the differential invariant given by (3.1), taking into account (3.2)–(3.3), we get

$$\left(-2\frac{q_1}{q_4} - q_4\right)\frac{\partial J}{\partial q_1} + \left(\frac{1}{2}q_1 - \frac{q_2 q_4}{q_1} + \frac{q_1 q_3}{q_4}\right)\frac{\partial J}{\partial q_2} + \frac{\partial J}{\partial q_3} - 2\frac{\partial J}{\partial q_4} = 0.$$

The corresponding characteristic equations give

$$J = J(r_1, r_2, r_3),$$

where

$$r_1 = \frac{f}{u_x} - f_{u_x} + \frac{1}{2}u_x f_{u_x u_x},$$
$$r_2 = f_u - \frac{1}{2}u_x f_{u u_x} + \frac{1}{2} f f_{u_x u_x} - \frac{1}{2}u_x f_{u_x} f_{u_x u_x} + \frac{1}{4}u_x^2 f_{u_x u_x}^2,$$
$$r_3 = \frac{f - u_t}{u_x} - \frac{1}{2}u_x f_{u_x u_x}.$$

Finally, the invariant test

$$Y_1^{(2)}(J) = 0,$$

after some calculations, yields

$$r_1\frac{\partial J}{\partial r_1} + 2r_2\frac{\partial J}{\partial r_2} + r_3\frac{\partial J}{\partial r_3} = 0.$$

From the corresponding characteristic equations, provided that

$$2f - 2u_t - u_x^2 f_{u_x u_x} \neq 0,$$

we get that the general form of second-order differential invariants of equation (1.1) is

$$J = J(\lambda_1, \lambda_2),$$

with λ_1 and λ_2 given by

$$\lambda_1 = \frac{2f - 2u_x f_{u_x} + u_x^2 f_{u_x u_x}}{2f - 2u_t - u_x^2 f_{u_x u_x}}, \tag{3.4}$$

$$\lambda_2 = \frac{\left(4f_u - 2u_x f_{u u_x} + 2 f f_{u_x u_x} - 2u_x f_{u_x} f_{u_x u_x} + u_x^2 f_{u_x u_x}^2\right)u_x^2}{\left(2f - 2u_t - u_x^2 f_{u_x u_x}\right)^2}. \tag{3.5}$$

4. Some applications

In some previous papers [8, 9] the differential invariants for the family of equations $u_{tt} - u_{xx} = f(u, u_t, u_x)$ have been obtained and applied in order to characterize some linearizable subclasses of those equations.

Similar techniques have also been used in [10] to bring linear parabolic equations to the classical heat equation.

Here we wish to use the second-order invariants λ_1 and λ_2 in order to bring nonlinear equations of the class (1.1) in linear form by using equivalence transformations of the admitted group $G_{\mathcal{E}}$.

The search for transformations mapping a nonlinear differential equation in a linear differential equation has interested several authors. In particular Kumei and Bluman in their pionering paper [11] gave some necessary and sufficient conditions that, by examining the invariance algebra, allow us to affirm whether a nonlinear equation is transformable in the linear form.

It is worthwhile to note that the Kumei–Bluman algorithm (see also [12]) constructing the linearizing map, based on the existence of an admitted infinite parameter Lie group transformations, does not require knowledge, *a priori*, of a specific linear target equation. The target of the transformation comes out in a natural way during the developments of the algorithm. Here, instead, we search the nonlinear equations of the class (1.1) that can be mapped by an equivalence transformation in a linear equation of the subclass

$$v_\tau - v_{\sigma\sigma} = k_0 v_\sigma, \tag{4.1}$$

with $k_0 = \text{const}$.

That is, once fixed *a priori* the target (4.1) we characterize the whole set of equations (1.1) which can be mapped in (4.1).

For the subclass (4.1) the differential invariants λ_1 and λ_2 are zero. So, taking into account (3.4), (3.5), we search the functional forms of $f(u, u_x)$ for which

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0. \end{cases}$$

Then, solving

$$2f - 2u_x f_{u_x} + u_x^2 f_{u_x u_x} = 0, \tag{4.2}$$

we get

$$f = u_x^2 h(u) + h_1(u) u_x$$

where h and h_1 are arbitrary functions of u.

By requiring that

$$\lambda_2|_{f=u_x^2 h(u)+h_1(u)u_x} = 0, \tag{4.3}$$

we get

$$h_1(u) = h_0$$

where h_0 is a constant.

We are able, now, to affirm:

Theorem 1. *An equation belonging to the class (1.1) can be transformed in a linear equation of the form (4.1), by an equivalence transformation of $L_\mathcal{E}$, if and only if the function f is given by*

$$f = u_x^2 h(u) + h_0 u_x. \tag{4.4}$$

Proof. From equations (4.2) and (4.3) it follows that the condition (4.4) is necessary.

In order to demonstrate that it is sufficient, we must show that there exists at least an ET transforming the equations

$$u_t - u_{xx} = u_x^2 h(u) + h_0 u_x \tag{4.5}$$

in (4.1).

The finite form of the equivalence transformations generated by (2.4)–(2.6) is

$$t = \tau\,\mathrm{e}^{-\varepsilon_1} - \varepsilon_0, \qquad x = (\sigma - \tau\varepsilon_2 - \varepsilon_3)\,\mathrm{e}^{-\frac{1}{2}\varepsilon_1}, \qquad u = \psi(v), \tag{4.6}$$

where ψ is an arbitrary function, with $\psi'(v) \neq 0$, and ε_i are arbitrary parameters.

By applying the transformation (4.6) to equations (4.5), we get

$$v_\tau - v_{\sigma\sigma} = v_\sigma^2 \frac{\psi'^2 h(\psi(v)) + \psi''}{\psi'} + \left(h_0\,\mathrm{e}^{-\frac{1}{2}\varepsilon_1} - \varepsilon_2\right) v_\sigma. \tag{4.7}$$

By choosing as $\psi(v)$ a solution of ODE

$$\frac{\psi'^2 h(\psi(v)) + \psi''}{\psi'} = 0,$$

the transformed equation (4.7) takes the linear form (4.1) where $k_0 = h_0\,\mathrm{e}^{-\frac{1}{2}\varepsilon_1} - \varepsilon_2$. □

It is a simple matter to show that it is possible to choose the arbitrary parameters ε_i in order to make $k_0 = 0$. So we can affirm, by assuming, for sake of simplicity, $\varepsilon_1 = 0$ and $\varepsilon_2 = h_0$:

Corollary 1. *The group of ETs*

$$t = \tau - \varepsilon_0, \qquad x = \sigma - \tau h_0 - \varepsilon_3, \qquad u = H^{-1}(c_0 v + c_1), \tag{4.8}$$

with H^{-1} denoting the inverse function of $H(\psi) = \int^\psi \mathrm{e}^{\int^w h(z)\mathrm{d}z}\mathrm{d}w$ and with $c_0 \neq 0, c_1$ arbitrary constants, maps the equations of the form (4.5) in the equation

$$v_\tau - v_{\sigma\sigma} = 0.$$

Example 1. We consider the equation

$$u_t - u_{xx} = -u_x^2 tgu + u_x. \tag{4.9}$$

In this case $h(u) = -tgu$ and $h_0 = 1$, so $H(\psi) = \sin\psi$ and the transformations (4.8) become

$$t = \tau - \varepsilon_0, \qquad x = \sigma - \tau - \varepsilon_3, \qquad u = \arcsin(c_0 v + c_1). \tag{4.10}$$

It is a simple matter to verify that the transformation (4.10) maps equation (4.9) in

$$v_\tau - v_{\sigma\sigma} = 0.$$

Remark. A special case of equation (4.5) is the Burger's equation in the potential form

$$u_t - u_{xx} = u_x^2. \tag{4.11}$$

One can ascertain that the transformation (4.8.III) for (4.11) becomes the well-known Cole–Hopf transformation which maps the considered equations in the well-studied linear Fourier's equation

$$w_t - w_{xx} = 0.$$

5. Conclusions

In this paper we considered a family of semilinear diffusion equations and following [2, 3] we have obtained the differential invariants of second order with respect to equivalence transformations for this family by the infinitesimal method.

As an application, we have proved that a family of generalized diffusion equations can be reduced to the heat equation

$$v_\tau - v_{\sigma\sigma} = 0$$

via appropriate equivalence transformations.

Finally, for the Burger's equation in potential form from the transformation (4.8) we recover the well-known Cole–Hopf transformation which brings it to the heat equation.

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